

Re-Reconstruction the Proof of Babylonian Volume Formula for The Frustum of a Square Pyramid

Lingzhi Li and Wenhua Sun

Department of Mathematics, Shandong University of Technology, Zibo 255000, China

Abstract

Ivor Grattan-Guinness and Jiwei Zhao have already given the explanations about the volume formula for the frustum of a square pyramid of the ancient Babylonian. By means of the segmentation method, this paper gives a new reconstruction of ancient proof about the formula, which is simpler than the above. The reconstruction of the proof of this volume formula can benefit the mathematical education, at the same time it may deepen the understanding of Babylonian geometrical thoughts.

Keywords

Ancient Babylonian Volume formula; frustum of a square pyramid; Segmentation method.

1. Introduction

There are maybe more than ten kinds of methods to calculate the volume for the frustum of a square pyramid in the literature, please see the references [1-4] and cited in there. However, a very interesting thing is the ancient Babylonian formula which is recorded in the Tablet Writing as the following ecctetric forms

$$V = \left[\left(\frac{b+a}{2} \right)^2 + \frac{1}{3} \left(\frac{b-a}{2} \right)^2 \right] h \quad (1)$$

Where a, b are the side-length of the bottom surface, and h is the height of the frustum of a square pyramid respectively.

Why and how did the ancient Babylonians get such a puzzling formula? Its a interting question. The paper Reconstruction the Proof of Babylonian Volume Formula for The Frustum of a Square Pyramid gives an explanation about the formula: put the regular square prism and the frustum of a square pyramid together to make sure the intermediate planes coincide, then resolve it into the volumes of a regular prism and special upright slices, and eventually derived Babylonian volume formula for the frustum of a square pyramid. Based on the above method, we obtain a new restoration way which is better, we think, than the previous methods given by [4].

2. Re-Reconstruction of ancient proofs

2.1. Quarter Segmentation of the Frustum of A Square Pyramid

To get the formula (1), Firstly, we make a regular square prism whose bottom surface is the same as the intermediate plane of the frustum of a square pyramid, and its side-length of the bottom surface is $\frac{a+b}{2}$ and the height is h , as shown in Figure 1. Then divide it into four same

parts by using a plane which is perpendicular to the bottom of the frustum of a square pyramid, we can find the volume formula $V_A = 4 \times V_B$, as shown in Figure 2.

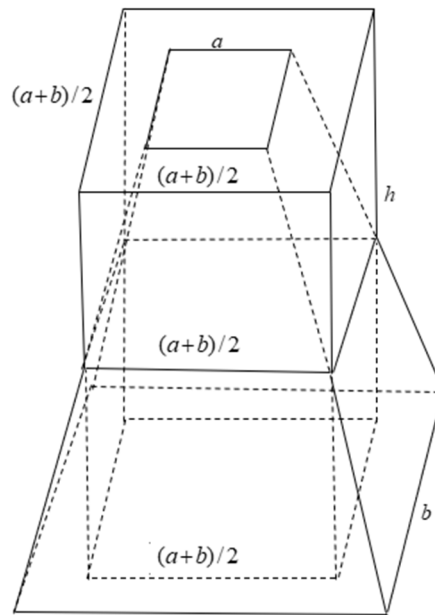


Figure 1. A

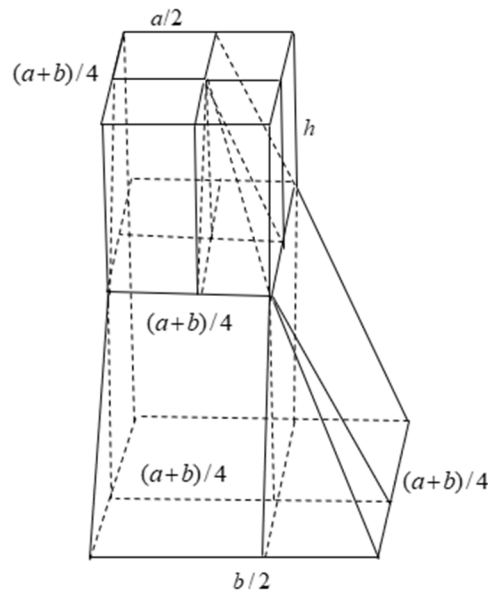


Figure 2. B

Then divide B into two parts by using the intermediate plane, C is the part below the intermediate plane and D is the remaining part. We can find the volume formula as

$$V_B = V_C + V_D \quad (2)$$

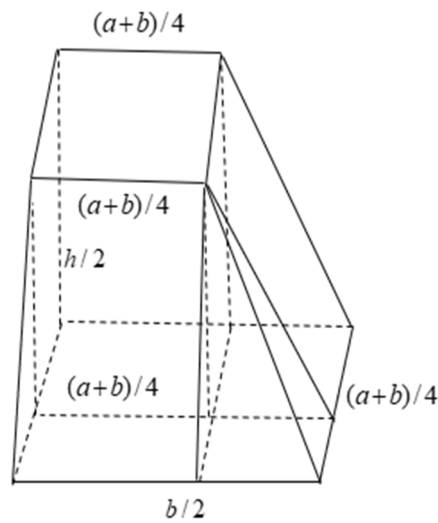


Figure 3. C

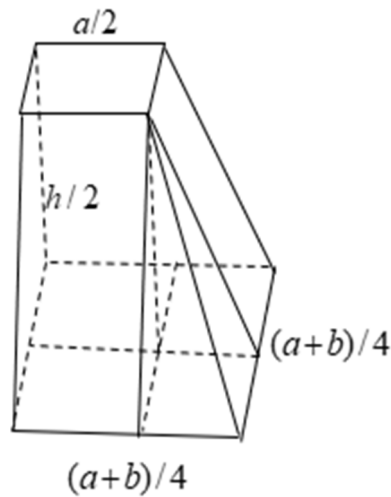


Figure 4. D

2.2. Calculation of separated Pyramid

In this section, we try to calculate the volumes separated parts of C and D. In view of the difficulty to get the volume of C, we divide C into four parts: C_1 , C_2 , C_3 and C_4 , as shown in Figure 5-8, then the volume formula of C can be calculated as

$$\begin{aligned}
 C &= C_1 + C_2 + C_3 + C_4 \\
 &= \left(\frac{a+b}{4}\right)^2 \frac{h}{2} + \frac{1}{2} \cdot \frac{h}{2} \cdot \frac{b-a}{2} \cdot \frac{a+b}{4} \\
 &\quad + \frac{1}{2} \cdot \frac{h}{2} \cdot \frac{b-a}{4} \cdot \frac{a+b}{4} + \frac{1}{3} \left(\frac{b-a}{4}\right)^2 \frac{h}{2} \\
 &= \left(\frac{a+b}{4}\right)^2 \frac{h}{2} + \frac{h}{2} \cdot \frac{b-a}{4} \cdot \frac{a+b}{4} + \frac{1}{3} \left(\frac{b-a}{4}\right)^2 \frac{h}{2}.
 \end{aligned} \tag{3}$$

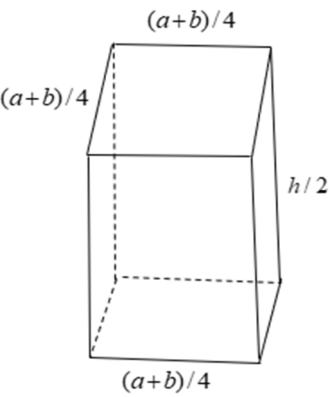


Figure 5. C_1

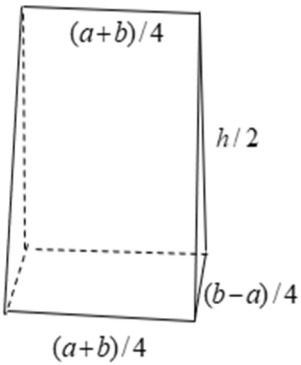


Figure 6. C_2

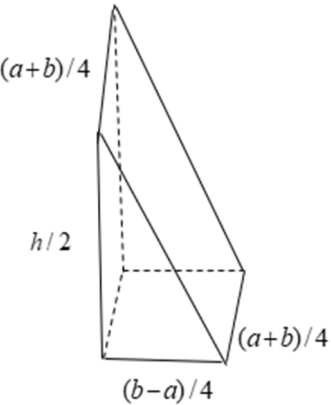


Figure 7. C_3

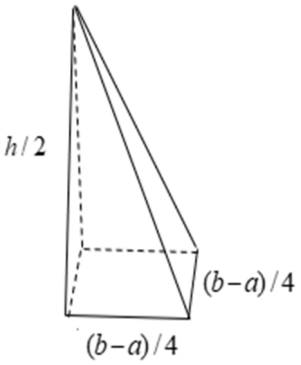


Figure 8. C_4

Due to the volume of the D includes parts that do not belong to the A, by dividing D into four parts D_1, D_2, D_3 and D_4 , as shown in Figure 9-12, we can get the volume formula as

$$\begin{aligned}
 D &= D_1 - D_2 - D_3 - D_4 \\
 &= \left(\frac{a+b}{4} \right)^2 \frac{h}{2} - \frac{1}{2} \cdot \frac{b-a}{4} \cdot \frac{h}{2} \cdot \frac{a}{2} \\
 &\quad - \frac{1}{2} \cdot \frac{b-a}{4} \cdot \frac{h}{2} \cdot \frac{a}{2} - \left[\left(\frac{b-a}{4} \right)^2 \frac{h}{2} - \frac{1}{3} \left(\frac{b-a}{4} \right)^2 \frac{h}{2} \right] \\
 &= \left(\frac{a+b}{4} \right)^2 \frac{h}{2} - \frac{b-a}{4} \cdot \frac{h}{2} \cdot \frac{a}{2} - \frac{2}{3} \cdot \left(\frac{b-a}{4} \right)^2 \cdot \frac{h}{2}
 \end{aligned} \tag{4}$$

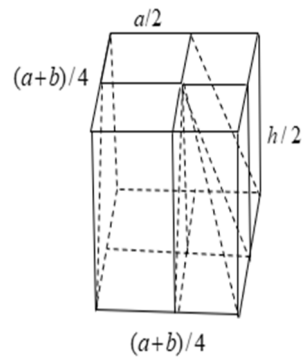


Figure 9. D_1

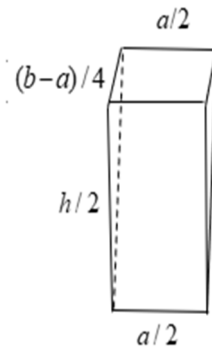


Figure 10. D_2

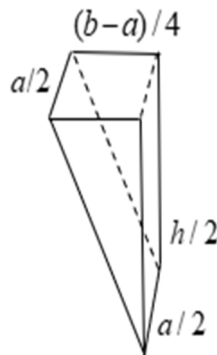
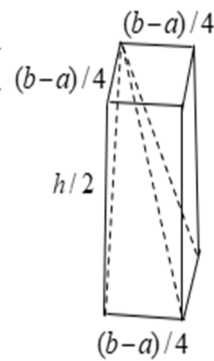


Figure 11. D_3

**Figure 12.** D_4

By the above calculation, the volume of B can be shown as

$$\begin{aligned}
 B = C + D &= \left[\left(\frac{a+b}{4} \right)^2 \frac{h}{2} + \frac{b-a}{4} \cdot \frac{h}{2} \cdot \frac{a+b}{4} + \frac{1}{3} \left(\frac{b-a}{4} \right)^2 \frac{h}{2} \right] \\
 &\quad + \left[\left(\frac{a+b}{4} \right)^2 \frac{h}{2} - \frac{b-a}{4} \cdot \frac{h}{2} \cdot \frac{a}{4} - \frac{2}{3} \left(\frac{b-a}{4} \right)^2 \frac{h}{2} \right] \\
 &= \left(\frac{a+b}{4} \right)^2 h + \frac{b-a}{4} \cdot \frac{h}{2} \cdot \left(\frac{a+b}{4} - \frac{a}{2} \right) - \frac{1}{3} \left(\frac{b-a}{4} \right)^2 \cdot \frac{h}{2} \\
 &= \left(\frac{a+b}{4} \right)^2 \cdot h + \frac{1}{3} \left(\frac{b-a}{4} \right)^2 \cdot h
 \end{aligned} \tag{5}$$

Then we get the volume of A as

$$\begin{aligned}
 A &= 4 \cdot B \\
 &= 4 \cdot \left(\frac{a+b}{4} \right)^2 \cdot h + \frac{4}{3} \cdot \left(\frac{b-a}{4} \right)^2 \cdot h \\
 &= \left[\left(\frac{b+a}{2} \right)^2 + \frac{1}{3} \cdot \left(\frac{b-a}{2} \right)^2 \right] \cdot h,
 \end{aligned} \tag{6}$$

Which is amazing formula just as shown by the ancient Babylonian in the Tablet Writing.

3. Conclusion

We finally derived the ancient Babylonian volume formula for the frustum of a square pyramid (1), which maybe there exist many other proof methods, but to find a way to prove it is a very meaningful thing in itself.

Acknowledgements

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